

Homework 9

Math 121A, Fall 2024.

Due: Monday, October 27, 11:59PM (completion only)

Unless otherwise stated, all solutions require clear, well-written justification. You must explicitly credit any collaborators for each problem.

Problem 1. By last homework, the singularities of $\frac{1}{\sin z}$ are all at the zeros of $\sin(z)$, which occur only at $z_0 = \pi k, k \in \mathbb{Z}$. Each one is a simple pole, which will be verified when we compute the residue:

$$\lim_{z \rightarrow k\pi} (z - k\pi) \frac{1}{\sin(z)} = \frac{1}{(\cos(k\pi))} = \begin{cases} 1 & k \text{ even} \\ -1 & k \text{ odd.} \end{cases}$$

Now for the integrals. The first encloses only one pole at 0, so

$$\int_{|z|=1} f(z) dz = 2\pi i \text{Res}_0 f = 2\pi i.$$

The second encloses simple poles at $-\pi, 0, \pi$, and no other poles, since $2\pi > 5$:

$$\int_{|z|=5} f(z) dz = 2\pi i (\text{Res}_{-\pi} f + \text{Res}_0 f + \text{Res}_{\pi} f) = 2\pi i (-1 + 1 - 1) = -2\pi i.$$

The final one finds all $2 < \pi k < 6$, which is just π :

$$\int_{|z-4|=2} f(z) dz = 2\pi i \text{Res}_{-\pi} f = -2\pi i.$$

Problem 2. (*The Cauchy distribution*) Observe that

$$f(z) = \frac{1}{\pi(1+z^2)} = \frac{1}{\pi(z+i)(z-i)}.$$

So f has simple poles at i and $-i$. We set up a contour in the upper half plane, oriented positively, with C the upper semicircle of radius R , so that by the residue theorem,

$$\int_C f(z) dz + \int_{-R}^R f(x) dx = 2\pi i \text{Res}_i f = 2\pi i (-i/2\pi),$$

where we have computed the residue in the usual way. Now, we estimate the semicircular integral and

show that it vanishes as $R \rightarrow \infty$:

$$\begin{aligned} \left| \int_C f(z) dz \right| &\leq \int_C |f(z)| |dz| \\ &= \int_0^\pi \frac{|iRe^{i\theta}|}{\pi|1 + (Re^{i\theta})^2|} d\theta \\ &\approx \int_0^\pi \frac{R}{\pi R^2} d\theta \\ &= \frac{1}{R} \rightarrow 0. \end{aligned}$$

where in the third line we have used that when R is very large, $|1 + (Re^{i\theta})^2| \approx |(Re^{i\theta})^2| = R^2$, and in the last line, we have taken $R \rightarrow \infty$. This means that this integral vanishes, and so we are left with

$$\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} dx = 2\pi i(-i/2\pi) = 1,$$

verifying that this indeed is a probability distribution.

Problem 3. Compute the following integrals via contour integration. (The solutions are all at the end of this document.)

- (a) $\int_0^{2\pi} \frac{1}{5-3\cos\theta} d\theta.$
- (b) $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta.$
- (c) $\int_0^\infty \frac{1}{(4x^2+1)^3} dx.$
- (d) $\int_0^\infty \frac{x^2}{(x^2+4)(x^2+9)} dx.$
- (e) $\int_{-\infty}^\infty \frac{x \sin x}{x^2+4x+5} dx.$
- (f) $\int_0^\infty \frac{\cos x}{(1+9x^2)^2} dx.$
- (g) $\int_0^\infty \frac{1}{1+x^6} dx.$
- (h) Boas problem 14.7.37a)

Problem 4. (*Fourier Series*)

Let $f(z) = \sum_{n=0}^\infty a_n z^n$ converge for $|z| < R$.

- (a) Pick an arbitrary index n . f is holomorphic on its radius of convergence, and so we may apply Cauchy's integral formula for the n^{th} derivative to a circle γ_r of radius r for any $0 < r < R$, to obtain that

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(re^{i\theta})}{(re^{i\theta} - 0)^{n+1}} ire^{i\theta} d\theta = \frac{n!}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta.$$

Then, by Taylor's theorem, $a_n = \frac{f^{(n)}(0)}{n!}$, so we have

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta.$$

The most common form of this is where $r = 1$, the unit circle.

(b) Following the hint,

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \overline{f(re^{i\theta})} f(re^{i\theta}) d\theta \\
&= \frac{1}{2\pi} \sum_{n,k=0}^{\infty} \int_0^{2\pi} \overline{a_n r^n e^{in\theta}} a_k r^k e^{ik\theta} d\theta \\
&= \frac{1}{2\pi} \sum_{n=0}^{\infty} 2\pi \overline{a_n} a_n r^{2n} \\
&= \sum_{n=0}^{\infty} |a_n|^2 r^{2n},
\end{aligned}$$

where the crucial third line step holds because r is real and

$$\int_0^{2\pi} e^{-in\theta} e^{ik\theta} d\theta = \int_0^{2\pi} e^{i(k-n)\theta} d\theta = \begin{cases} 0 & k \neq n \\ 2\pi & k = n. \end{cases}$$

This last condition is the most essential part of the proof! This means that the functions $e^{in\theta}$ are *orthogonal* with respect to the L^2 inner product on functions $\langle f, g \rangle = \int_0^{2\pi} \overline{f(\theta)} g(\theta) d\theta$.

Problem 5. (*Argument principle*)

Let f be meromorphic on a region Ω . Consider the “logarithmic derivative” $F(z) = f'(z)/f(z)$.

(a) Let z_0 be an n^{th} order zero of $f(z)$. Following the hint, we use the local description of zeros:

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \dots$$

Then,

$$F(z) = \frac{f'(z)}{f(z)} = \frac{na_n(z - z_0)^{n-1} + O((z - z_0)^n)}{a_n(z - z_0)^n + O((z - z_0)^{n+1})}.$$

Let us compute the residue. It will be a simple pole, as evidenced by the below limit existing:

$$\lim_{z \rightarrow z_0} (z - z_0) F(z) = \lim_{z \rightarrow z_0} \frac{na_n(z - z_0)^n + O((z - z_0)^{n+1})}{a_n(z - z_0)^n + O((z - z_0)^{n+1})} = n.$$

By the residue theorem, the integral over any simple loop γ containing z_0 and no other zeros or poles oriented positively is

$$\int_{\gamma} F(z) dz = \int_{\gamma} \frac{f'(z)}{f(z)} dz = 2\pi i n.$$

(b) Similar to the above approach, we use the local description of f near a pole of order p :

$$f(z) = \frac{a_{-p}}{(z - z_0)^p} + \dots + \frac{a_{-1}}{z - z_0} + g(z),$$

where $g(z)$ is holomorphic near z_0 . Then,

$$F(z) = \frac{f'(z)}{f(z)} = \frac{(z - z_0)^p f'(z)}{(z - z_0)^p f(z)} = \frac{-pa_{-p}/(z - z_0) + O(1)}{a_{-p} + O(1)},$$

whose residue may be computed to be

$$\text{Res}_{z_0} F(z) = \lim_{z \rightarrow z_0} (z - z_0) F(z) = -p.$$

Again by the residue theorem we have

$$\int_F (z) dz = \int_\gamma \frac{f'(z)}{f(z)} dz = -2\pi ip.$$

- (c) Putting the two parts together and using the residue theorem, if γ encircles N zeros of f (counting multiplicity) and P poles of f (counting multiplicity), we have

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i(N - P).$$

We can use arguments similar to that of the residue theorem to now find zeros and poles! This is extremely useful in control theory and circuit design, among other applications.

Problem 3

a) Substitute $z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta = iz d\theta$

$$\cos \theta = \frac{z + z^{-1}}{2}$$

$$\text{So } I = \int_0^{2\pi} \frac{d\theta}{5 - 3\cos \theta} = \oint_{|z|=1} \frac{1}{iz} \frac{dz}{5 - \frac{3}{2}(z + \frac{1}{z})}$$

$$= -i \oint \frac{1}{z} \frac{dz}{\frac{10z - 3z^2 - 3}{2z}} = 2i \oint \frac{dz}{-(10z - 3z^2 - 3)}$$

$$= 2i \oint \frac{dz}{(3z-1)(z-3)}$$

Poles at $z = \frac{1}{3}$ and $z = 3$.

$$\begin{aligned} \text{Res}(\frac{1}{3}) &= \lim_{z \rightarrow \frac{1}{3}} (z - \frac{1}{3}) \frac{1}{(3z-1)(z-3)} = \frac{1}{3(\frac{1}{3}-3)} \\ &= -\frac{1}{8} // \end{aligned}$$

$$\text{So } I = \cancel{2\pi i} \cdot \cancel{2i} \cdot \left(-\frac{1}{8}\right) = \frac{\pi}{2} //$$

b) As in the previous problem, we take $z = e^{i\theta}$
 so $\cos\theta = \frac{1}{2}(z + z^{-1})$ and $\cos 2\theta = \frac{1}{2}(z^2 + z^{-2})$. Then

$$I = \oint_{|z|=1} \frac{1}{iz} \frac{\frac{1}{2}(z^2 + \frac{1}{z^2})}{5 + 4\frac{1}{2}(z + \frac{1}{z})} dz$$

$$= -i \oint \frac{\frac{z^4+1}{z^2}}{10z + 4z^2 + 4} dz = \frac{-i}{4} \oint \frac{z^4+1}{z^2(z+2)(z+1/2)} dz$$

Singularities at $z=0, -2, -1/2$.

$$\text{Res}(0) = \lim_{z \rightarrow 0} \frac{d}{dz} \cancel{z} \cdot \frac{(z^4+1)}{\cancel{z^2}(z+2)(z+1/2)}$$

$$= \lim_{z \rightarrow 0} \frac{(4z^3)}{(z+2)(z+1/2)} + \frac{(z^4+1)}{(z+2)} \frac{(-1)}{(z+1/2)^2} + \frac{(z^4+1)}{(z+1/2)} \frac{(-1)}{(z+2)^2}$$

$$= \frac{-1}{2 \cdot \frac{1}{4}} - \frac{1}{\frac{1}{2} \cdot 4} = -2 - \frac{1}{2} = \underline{\underline{-\frac{5}{2}}}$$

$$\text{Res}(-1/2) = \lim_{z \rightarrow -1/2} \cancel{(z+1/2)} \frac{(z^4+1)}{(z+2)\cancel{(z+1/2)}z^2} = \frac{\frac{1}{16} + 1}{3/2 \cdot \frac{1}{4}}$$

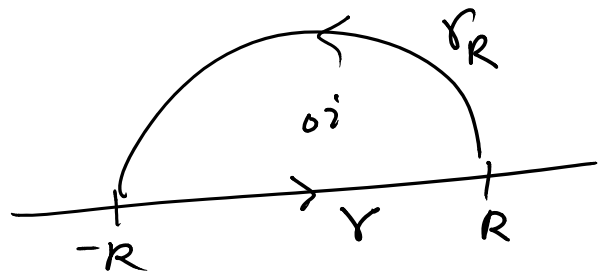
$$= \frac{17}{16} \cdot \frac{8}{3} = \underline{\underline{\frac{17}{6}}}$$

$$\text{So } I = 2\pi i \left(-\frac{5}{2}\right) \left(-\frac{5}{2} + \frac{17}{6}\right) = \frac{2}{6} \cdot \frac{\pi}{2} = \pi/6 //$$

c) $2I = \int_{-\infty}^{\infty} \frac{dx}{(4x^2+1)^3}$ Since the function is even.

let $f(z) = \frac{1}{(4z^2+1)^3} = \frac{1}{64 (z+\frac{i}{2})^3 (z-\frac{i}{2})^3}$
 Singularities at $z = \pm \frac{i}{2}$

Consider $C_R = \gamma + \gamma_R$



By the Residue theorem:

$$\oint_{C_R} f(z) dz = 2\pi i \operatorname{Res}\left(\frac{i}{2}\right)$$

$$\begin{aligned} \operatorname{Res}(i) &= \frac{1}{2!} \lim_{z \rightarrow i/2} \frac{d^2}{dz^2} \frac{1}{64} \cancel{(z-i/2)^3} \frac{1}{(z+i/2)^3 \cancel{(z-i/2)^2}} \\ &= \frac{1}{2!} \frac{1}{64} \frac{(-3)(-4)}{(i/2+i/2)^5} = \frac{12}{128} \frac{1}{i} = -\frac{3i}{32} \end{aligned}$$

$$\text{So } \oint_{C_R} f(z) dz = \int_{-R}^R f(x) dx + \int_{\gamma_R} f(z) dz = \frac{-6\pi i^2}{32} = \underline{\underline{\frac{3\pi}{16}}}$$

where $\gamma_R(t) = Re^{it} \quad t \in [0, \pi]$

By the triangle inequality, $\int_{\gamma_R} f(z) dz \leq \left(\max_{z \in \gamma_R} |f(z)| \right) \pi R$

Since $|z|=R$ on γ_R , we have

$$|f(z)| = \frac{1}{|4z^2+1|^3} \leq \frac{1}{(4|z|^2-1)^3} = \frac{1}{(4R^2-1)^3} = O\left(\frac{1}{R^6}\right)$$

So $\int_{\gamma_R} f(z) dz = O\left(\frac{1}{R^5}\right) \rightarrow 0$ as $R \rightarrow \infty$

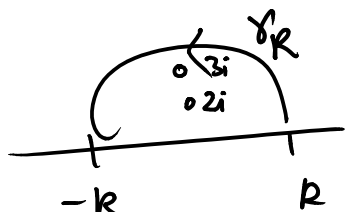
and $2I = \lim_{R \rightarrow \infty} \oint_{\gamma_R} f(z) dz = \frac{3\pi}{16} \Rightarrow I = \underline{\underline{\frac{3\pi}{32}}}$

8)

$$2I = \int_{-\infty}^{\infty} \frac{z^2}{(z^2+4)(z^2+9)} dz \quad \text{because even.}$$

$$\text{Let } f(z) = \frac{z^2}{(z^2+4)(z^2+9)} = \frac{z^2}{(z+2i)(z-2i)(z+3i)(z-3i)}$$

There are no singularities on $\mathbb{R}$ so we use the same semicircle as in the last question:

$$\oint_{\gamma_R} f(z) dz = 2\pi i (\text{Res}(2i) + \text{Res}(3i))$$


These are simple poles so

$$\text{Res}(2i) = \lim_{z \rightarrow 2i} \frac{z^2}{(z+2i)(z-3i)(z+3i)}$$

$$= \frac{4i^2}{(4i)(-i)(5i)} = \frac{-4}{20i} = \frac{i}{5}$$

$$\text{Res}(3i) = \lim_{z \rightarrow 3i} \frac{z^2}{(z+2i)(z-2i)(z+3i)}$$

$$= \frac{9i^2}{(5i)(i)(6i)} = \frac{-9}{-30i} = -\frac{3}{10}i$$

$$\text{So } \oint_{C_R} = 2\pi i \cdot i \left(\frac{1}{5} - \frac{3}{10} \right) = \frac{2\pi}{10} = \frac{\pi}{5} //$$

As before, we have $\max_{z \in \gamma_R} |f(z)| \leq \frac{|z|^2}{(|z|^2-4)(|z|^2-9)}$

So by the triangle inequality, $\int_{\gamma_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$. $= O\left(\frac{1}{R^2}\right)$

Thus $I = \frac{\pi}{10}$

e)

$$I = \int_{-\infty}^{\infty} \frac{x \sin x \, dx}{x^2 + 4x + 5}. \text{ Take } f(z) = \frac{z e^{iz}}{(z+2-i)(z+2+i)}$$

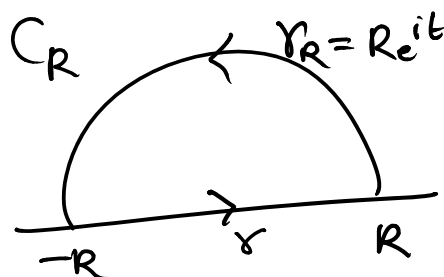
with simple poles at $-2 \pm i$.

$$x = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$$

$$\text{Then } I = \operatorname{Im} \left(\int_{-\infty}^{\infty} \frac{z e^{iz}}{(z+2-i)(z+2+i)} dz \right).$$

There are no zeros on $\mathbb{R}$ and the integrand is of type $\frac{P(z) e^{iaz}}{Q(z)}$, $a > 0$ so we use a semicircle

Contour in the upper halfplane.



There is one pole in the upper halfplane:

$$\operatorname{Res}(-2+i) = \lim_{z \rightarrow -2+i} \frac{(\cancel{z+2-i}) z e^{iz}}{(\cancel{z+2-i})(z+2+i)}$$

$$= \frac{(-2+i) e^{-2i-1}}{(-\cancel{2+i} + \cancel{2+i})} =$$

$$= \frac{(-2+i)(\cos(-2) + i \sin(-2))}{e \cdot 2i}$$

$$= \frac{1}{e \cdot 2i} (-2\cos(2) + \sin(2) + i(\cos(2) + 2\sin(2)))$$

$$\text{So } \operatorname{Im}\left(\oint_{\gamma_R} f(z) dz\right) = \operatorname{Im}\left(\frac{\pi}{e} (-2\cos(2) + \sin(2) + i(\cos(2) + 2\sin(2)))\right)$$

$$= \frac{\pi}{e} (\cos(2) + 2\sin(2)) //$$

On the other hand, $\int_{\gamma_R} \frac{z e^{iz}}{z^2 + 4z + 5} dz \rightarrow 0$

by Jordan's lemma since $\deg(z) \leq \deg(z^2 + 4z + 5)$

$$\text{So } I = \frac{\pi}{e} (\cos(2) + 2\sin(2)). //$$

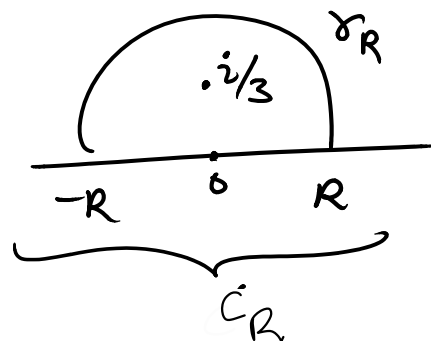
f)

$$2I = \int_{-\infty}^{\infty} \frac{\cos x}{(1+9x^2)^2} dx \quad \text{since integrand is even.}$$

Take $f(z) = \frac{1}{9^2} \frac{e^{iz}}{(z + \frac{i}{3})^2 (z - \frac{i}{3})^2}$. This has poles of order 2 at $z = \pm \frac{i}{3}$.

The Problem is now identical to the previous one, except we are interested in the real part (cosine).

Use a semicircle contour and Jordan's lemma to establish that



$$2I = \int_{-\infty}^{\infty} f(x) dx = \operatorname{Re} \left[\lim_{R \rightarrow \infty} \oint_{CR} f(z) dz \right] = \operatorname{Re} [2\pi i \operatorname{Res}(i/3)]$$

The residue is:

$$\operatorname{Res}(i/3) = \lim_{z \rightarrow i/3} \frac{d}{dz} \left(\cancel{z - i/3}^2 \right) \frac{1}{81} \frac{e^{iz}}{(\cancel{z - i/3})^2 (z + i/3)^2}$$

$$= \frac{1}{81} \lim_{z \rightarrow i/3} \frac{e^{iz} (-2)}{(z + i/3)^3} + \frac{ie^{iz}}{(z + i/3)^2}$$

$$= \frac{1}{81} \left(\frac{-2e^{-1/3}}{8i^3/27} + \frac{ie^{-1/3}}{4i^2/9} \right)$$

$$= \frac{-i}{9e^{1/3}} \cdot \quad \text{So } 2I = 2\pi i \frac{(-i)}{9e^{1/3}}$$

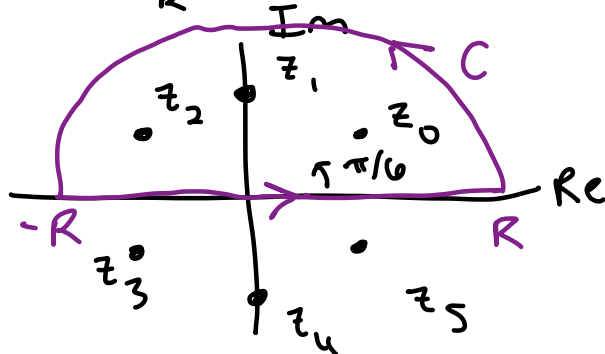
$$\Rightarrow I = \frac{\pi}{9e^{1/3}}$$

$$g) \int_0^{\infty} \frac{1}{1+x^6} dx.$$

By De Moivre, poles of $\frac{1}{1+z^6}$ are

$$z^6 = -1 = e^{i\pi}$$

$$\Rightarrow z_k = e^{i(\frac{\pi}{6} + \frac{2\pi}{6}k)}, \quad k = 0, 1, \dots, 5.$$



Choose semicircular contour.

$$\underbrace{\int_C \frac{1}{1+z^6} dz}_{\text{will show this goes to } 0} + \int_{-R}^R \frac{1}{1+x^6} dx = 2\pi i \left(\text{Res}_{z_0} \frac{1}{1+z^6} + \text{Res}_{z_1} \frac{1}{1+z^6} + \text{Res}_{z_2} \frac{1}{1+z^6} \right)$$

will show this goes to 0.

$$\left| \int_C \frac{1}{1+z^6} dz \right| \leq \int_C \frac{1}{|1+z^6|} |dz|$$

$$= \int_0^\pi \frac{|Re^{i\theta}|}{|1+Re^{i\theta}|^6} d\theta$$

Large
R limit

$$\rightarrow \approx \int_0^\pi \frac{R}{R^6} d\theta$$

$$= \frac{\pi}{R^5} \xrightarrow{R \rightarrow \infty} 0$$

Now for the residue. Each are simple poles:

$$\text{Res}_{z_0} \frac{1}{1+z^6} = \lim_{z \rightarrow z_0} (z - z_0) \cdot \frac{1}{1+z^6}$$

L'Hopital

$$\rightarrow = \lim_{z \rightarrow z_0} \frac{1}{6z^5}$$

$$= \frac{1}{6z_0^5}$$

(and similarly for the others)

$$\text{So}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6} dx = 2\pi i \left(\frac{1}{6} e^{-i5\pi/6} + \frac{1}{6} e^{-i5\pi/2} + \frac{1}{6} e^{-i25\pi/6} \right)$$

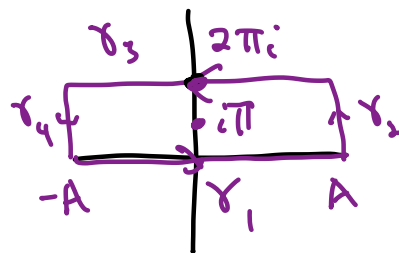
$$= \frac{2\pi}{3}$$

since this is even,

$$\left[\int_0^\infty \frac{1}{1+x^6} dx = \frac{\pi}{3} \right]$$

f) Boas 14.7, 37 a)

$$I = \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^x} dx, \text{ choose}$$



$$F(z) = \frac{e^{pz}}{1+e^z}, \quad 0 < p < 1.$$

$$\int_{\gamma} F(z) dz = 2\pi i \operatorname{Res}_{i\pi} F = 2\pi i \frac{e^{ip\pi}}{e^{i\pi}}$$

• Sidewalls: γ_2, γ_4

$$\cdot \text{ if } x \gg 1, \left| \frac{e^{p(x+iy)}}{1+e^{x+iy}} \right| \approx \left| \frac{e^{px}}{e^x} \right| = \left| \frac{1}{e^{(1-p)x}} \right| \rightarrow 0$$

$$\cdot \text{ if } x \ll -1, \left| \frac{e^{p(x+iy)}}{1+e^{x+iy}} \right| \approx \left| \frac{e^{-px}}{1} \right| \rightarrow 0$$

$$\text{So } \int_{\gamma_2 + \gamma_4} F(z) dz \rightarrow 0 \text{ as } A \rightarrow \infty$$

$$\int_{\gamma_3} F(z) dz = \int_A^{-A} \frac{e^{p(x+2\pi i)}}{1+e^{x+2\pi i}} dx = \left(- \int_{-A}^A \frac{e^{px}}{1+e^x} dx \right) e^{i2\pi p}$$

$$\Rightarrow (1 - e^{i2\pi p}) I = -2\pi i e^{i p \pi}$$

$$\Rightarrow \boxed{I = \frac{2\pi i}{e^{i\pi p} - e^{-i\pi p}} = \frac{\pi}{\sin p \pi}}$$